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Application of the Weighted Least Squares Parameter Estimation Method to the Robot Calibration

Significant attention has been paid recently to the topic of robot calibration. To improve the robot's accuracy, various approaches to the measurement of the robot's position and orientation (pose) and correction of its kinematic model have been proposed. Little attention, however, has been given to the method of estimation of the kinematic parameters from the measurement data. Typically, a least-squares solution method is used to estimate the corrections to the parameters of the model. In this paper, a method of kinematic parameter estimation is proposed where a standard least-squares estimation procedure is replaced by weighted least-squares. The weighting factors are calculated based on all the a priori available statistical information about the robot and the pose-measuring system. By giving greater weight to the measurements made where the standard deviation of the noise in the data is expected to be lower, a significant reduction in the error of the kinematic parameter estimates is made possible. The improvement in the calibration results was verified using a calibration simulation algorithm.

1 Introduction

To improve the robot's accuracy means to improve its ability to reach consistently a specified pose. Calibration of robots has been paid significant attention over the last few years as means of achieving better accuracy. Calibration improves accuracy by finding better estimates of the true parameters of the kinematic model used to control the robot's motion.

A calibration procedure involves measurement of the robot's pose at a number of locations, estimation of the kinematic model parameters, and application of the necessary corrections to the robot's controlling software. One of the most commonly used methods of kinematic parameter estimation relies on the fact that typically the deviation of the nominal parameters from the true value is not large. This means that the nonlinear relationship between the parameter space and task space can be linearized about the nominal values of the parameters. Such a model was derived by Wu [1] in 1984 and improved by Hayati and Mirmirani [2] in 1985. This model relates the errors in the robot's pose to errors in the kinematic parameters through a Jacobian-type matrix. Thus, the estimation of the kinematic parameter errors is reduced to the solution of the system of linear equations.

If $\mathbf{x}$ is a $(p \times 1)$ vector of p parameter errors and $\mathbf{y}$ is the $(N_m \times 1)$ vector containing N_m pose error measurements, then the linear model may be written as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \epsilon \tag{1}$$

where the $(N_m \times p)$ matrix $\mathbf{H}$ is the Generalized Jacobian matrix and ϵ is an $(N_m \times 1)$ vector of random deviations from the ideal model.

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There are always some error sources [as represented by ϵ in (1)] which cannot be included in the model. Among these are the unmodeled nongeometric errors [3] (e.g., backlash, gear-transmission errors, link compliance), random joint encoder errors and measurement errors. These errors will cause incorrect estimation of the kinematic parameters when the system in (1) is solved.

One way to reduce the effect of the unmolded errors is to take more measurements so that when the system is solved in a least-squares sense, a better estimate of the unknown parameters is obtained. However, increasing the number of measurements also raises the cost of calibration which limits the achievable estimate improvement.

In this paper, it is shown that, through the application of the Gauss-Markov Theorem (GMT) [4], it is possible to further improve the parameter estimates without increasing the number of measurements.

Gauss-Markov Theorem

Let

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \epsilon \tag{2}$$

be a linear model, where $\mathbf{y}$ is an $(N_m \times 1)$ observable random vector, $\mathbf{H}$ is an $(N_m \times p)$ known mapping matrix of rank $p < N_m$, $\mathbf{x}$ is a $(p \times 1)$ unknown nonrandom parameter vector, and ϵ is an $(N_m \times 1)$ vector whose elements are random such that

$$\mathbf{E}(\epsilon) = \mathbf{0} \quad \text{and} \quad \mathbf{E}(\epsilon\epsilon^T) = \mathbf{V}, \tag{3}$$

a known positive definite covariance matrix. Then the minimum variance linear unbiased estimator of $\mathbf{x}$, denoted by $\hat{\mathbf{x}}$, is given by

$$\hat{\mathbf{x}} = (\mathbf{H}^T\mathbf{V}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{V}^{-1}\mathbf{y} \tag{4}$$

whose variance $\mathbf{V}_{\hat{x}}$ is given by

$$\mathbf{V}_{\hat{x}}=(\mathbf{H}^T\mathbf{V}^{-1}\mathbf{H})^{-1} \tag{5}$$

If the rank of $\mathbf{H}$ is $r < p$, an extension of the GMT is possible based on the properties of the pseudoinverse of a matrix. Thus, in this case,

$$\hat{\mathbf{x}}=\mathbf{M}^+\mathbf{H}^T\mathbf{V}^{-1}\mathbf{y} \quad \mathbf{M}=\mathbf{H}^T\mathbf{V}^{-1}\mathbf{H} \tag{6}$$

where $\mathbf{M}^+$ designates the pseudoinverse of $\mathbf{M}$. The expressions in (4) and (6) imply that if the covariance matrix of the ϵ vector is known, then the best estimate is obtained by using the inverse of this covariance matrix as a weighting matrix. Thus, more weight would be given to the measurements with smaller variance. In this paper we will describe how the covariance matrix may be obtained in a calibration and then will demonstrate, through a computer simulation, the potential for estimate improvement.

2 Application of the Gauss-Markov Theorem to Calibration

In order to use the GMT for estimating the kinematic parameters, the $\mathbf{V}^{-1}$ weighting matrix has to be obtained. It contains the variances of the random component vector ϵ for each measurement point.

Mooring and Pack [5] determined that the randomness of the end effector positioning can be attributed to random variation in the robot's joint displacements. It was also found that the joint variable errors could be well approximated by normal distributions. Using this information, it is possible to derive the expression for the variance of the robot's pose measurements as a function of the joint variable values.

Let $\mathbf{n}_q$ be an $(N \times 1)$ vector (N being the number of joints of the robot) which represents the random noise in the joint variables. Let $\mathbf{n}_{mk}$ be a (6×1) vector of random noise due to measurement of the manipulator pose at point k. The elements of ϵ in (1) corresponding to measurement point k can be represented as

$$\epsilon_k=\mathbf{J}_k\mathbf{n}_q+\mathbf{n}_{mk} \tag{7}$$

where $\mathbf{J}_k$ is a $(6 \times N)$ Jacobian matrix of the manipulator at point k. Using (7), the vector ϵ can be assembled for N_p measurements,

$$\epsilon=\begin{bmatrix}\mathbf{J}_1\\ \mathbf{J}_2\\ \cdot\\ \cdot\\ \mathbf{J}_{N_P}\end{bmatrix}\mathbf{n}_q+\begin{bmatrix}\mathbf{n}_{m1}\\ \mathbf{n}_{m2}\\ \cdot\\ \cdot\\ \mathbf{n}_{mN_P}\end{bmatrix}=\mathbf{J}\ \mathbf{n}_q+\mathbf{n}_m \tag{8}$$

Assuming that $\mathbf{n}_q$ and $\mathbf{n}_m$ are independent random quantities, with $E(\mathbf{n}_q)=\mathbf{0}$ and $E(\mathbf{n}_m)=\mathbf{0}$, and with known covariance matrices

$$E(\mathbf{n}_q\mathbf{n}_q^T)=\Sigma_q \text{ and } E(\mathbf{n}_m\mathbf{n}_m^T)=\Sigma_m \tag{9}$$

the required covariance matrix $\mathbf{V}$ can be obtained by

$$\mathbf{V}=E(\epsilon\epsilon^T)=\mathbf{J}\ \Sigma_q\ \mathbf{J}^T+\Sigma_m \tag{10}$$

Then, either relation in (4) or (6) can be used to find $\hat{\mathbf{x}}$.

3 Simulation of the Calibration Process

A simulation of the calibration [6] was employed to find the improvement possible when an ordinary least-squares is replaced by a weighted least-squares solution in the estimation of the kinematic parameters. This simulation consists of the following steps:

(*1*) Specify N_p measurement points in the workspace of the robot by generating N_p sets of joint variable values:

$$\mathbf{q}_N^{(1)},\ \mathbf{q}_N^{(2)},\ \ldots,\ \mathbf{q}_N^{(N_p)} \tag{11}$$

where $\mathbf{q}_N^{(i)}$ is a $(N \times 1)$ vector of joint variable values for point i.

(*2*) Add geometric errors, $\Delta\mathbf{q}_G$ and $\Delta\mathbf{p}_G$, to the nominal kinematic model to obtain the simulated real robot's kinematic model. $\Delta\mathbf{q}_G$ is an $(N \times 1)$ vector of joint variable offsets and $\Delta\mathbf{p}_G$ is a $((p-N) \times 1)$ vector of errors in the constant kinematic parameters.

(*3*) For each point, perform a Forward Kinematic (FK) solution of the simulated real robot's kinematic model while adding the random joint variable error $\mathbf{n}_q$ to the joint variable values:

$$f(\mathbf{q}_N+\Delta\mathbf{q}_G+\mathbf{n}_q,\ \mathbf{p}_N+\Delta\mathbf{p}_G)=\mathbf{T}_R \tag{12}$$

where $\mathbf{p}_N$ is a $((p-N) \times 1)$ vector of constant kinematic parameters.

(*4*) Add measurement errors $\mathbf{n}_{mk}$ to $\mathbf{T}_R$ to determine simulated pose measurements, $\mathbf{T}_M$.

(*5*) Determine the pose errors $\mathbf{e}$ by finding the difference between $\mathbf{T}_M$ and $\mathbf{T}_N$ where $\mathbf{T}_N$ is such that

$$f(\mathbf{q}_N,\ \mathbf{p}_N)=\mathbf{T}_N \tag{13}$$

(*6*) Assemble the system in Eq. (1) from vectors $\mathbf{e}$ and matrices $\mathbf{H}$ for each point.

(*7*) Solve the assembled system by either ordinary or weighted least-squares method to estimate the kinematic parameters.

(*8*) Evaluate the parameter estimates by testing the accuracy of the estimated model within robot's workspace [7].

The simulation described above falls into the category of Monte Carlo simulations [8]. Normally distributed random numbers are added to the joint variable values and to the pose error values. The former simulates the finite repeatability of the robot and the latter, the finite precision of the pose-measuring device.

As a result of the test in the step 8 of the simulation procedure, for each simulation run, a value of the average workspace position error E_p and orientation error E_r are obtained. However, since these values are functions of random quantities in the simulation, they are random quantities as well. Thus, to measure the performance of a particular method, the simulation is repeated until a statistically significant sample is obtained.

4 Set-Up and Results of the Simulated Calibration

The kinematic model of PUMA 560 robot [9] was used for the simulation tests. The joint variable random errors were specified by the standard deviation of 0.007 degrees for joints 1–3 and 0.002 degrees for joints 4–6. These values resulted in a 0.1 mm mean position error at the end effector which equals the specified repeatability of the PUMA 560 [9].

The measurement points were specified by sets of joint variable values. These values were obtained by generating uniformly distributed random numbers within the joint travel limits. Only the position measurements were simulated and only the position performance of the simulated robot was tested.

A minimum of 10 points (3 measurements each) are required to estimate 30 unknown kinematic parameters. Tests comparing the performance of the weighted and ordinary least-squares solutions were performed for different numbers of measurement points, N_p, (ranging from 12 to 192), and different levels of measurement error standard deviation, σ_{me}, (ranging from 0 to 0.1 mm).

Based on the above simulation parameters, the covariance matrices Σ_q and Σ_m needed to calculate $\mathbf{V}$ were set as

$$\Sigma_q=diag(\sigma_1^2,\ \sigma_2^2,\ \ldots,\ \sigma_6^2) \tag{14}$$

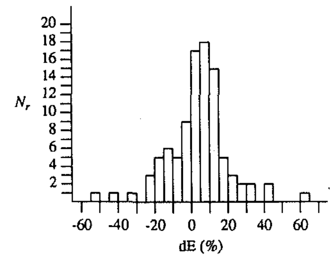

Fig. 1(*a*) *dE* distribution: 12 measurement points

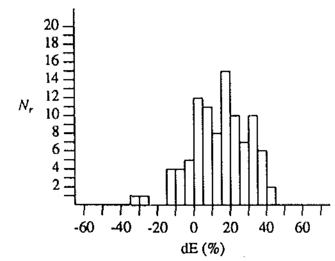

Fig. 1(*b*) *dE* distribution: 24 measurement points

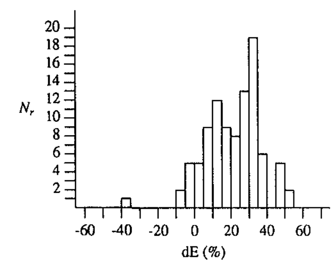

Fig. 1(*c*) *dE* distribution: 48 measurement points

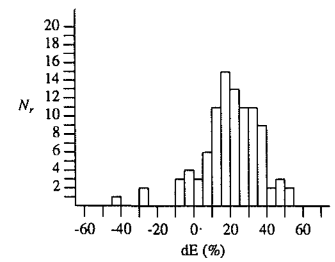

Fig. 1(*d*) *dE* distribution: 96 measurement points

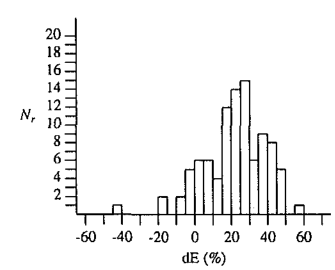

Fig. 1(*e*) *dE* distribution: 192 measurement points

where

$$\sigma_1 = \sigma_2 = \sigma_3 = 0.007 \ deg$$
$$\sigma_4 = \sigma_5 = \sigma_6 = 0.002 \ deg \tag{15}$$

and

$$\Sigma_m = diag(\sigma_{me}^2, \sigma_{me}^2, \ldots) \tag{16}$$

where Σ_m has the dimensions of $(3N_p \times 3N_p)$. To avoid numerical problems, the actual weighting matrix $\mathbf{W}$ used in the computations was obtained by normalizing $\mathbf{V}$ as

Table 1 Improvement (dE) as a function of $N_p \sigma_{me} = 0.0$)

N_r	MEAN	SD	LOWEST	HIGHEST	% ABOVE ZERO
12	3.97	16.31	-53.0	60.6	67.7
24	14.20	15.21	-31.6	42.8	84.4
48	21.24	15.16	-35.7	54.0	91.7
96	20.41	16.40	-44.3	53.7	89.6
192	21.71	17.16	-43.9	55.4	89.6

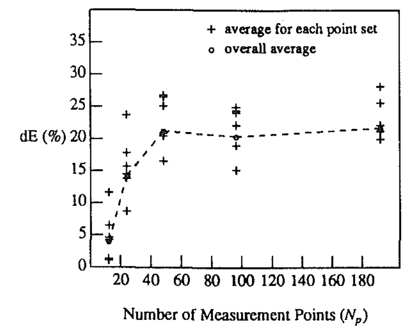

Fig. 2 Relative decrease in position error as a function of the member of measurement points

$$\mathbf{W} = \sigma_{\max}^2 \mathbf{V}^{-1} \tag{17}$$

where $\sigma_{\max}^2$ is the largest diagonal element of $\mathbf{V}$.

Six sets of N_p randomly chosen measurement points were tested for each value of N_p. For each set of measurement points, 16 runs of the simulation were repeated to obtain a statistically significant sample. The results of the comparison between the two least-squares solutions are presented as a relative decrease in the average position error after the application of the weighted least-squares. Thus, if E_p^w and E_p are the position error values as a result of the weighted and ordinary least-squares, respectively, then the results are given as

$$dE = 100 \times \frac{E_p - E_p^w}{E_p} \tag{18}$$

Therefore, a positive dE means that there was a decrease in the residual position error after the calibration due to the use of the weighted least-squares instead of the ordinary least-squares method.

Figure 1 shows the distribution of simulation runs (N_r) as a function of dE for different numbers of measurement points (with σ_{me} set to zero). Ninety-six samples obtained by performing 16 runs for 6 different sets of points make up the data used in each histogram. For each sample, E_p^w and E_p values were obtained by running the same simulation twice, first with the weighted and then with the ordinary least-squares solution. The same sequence of random numbers was used for both simulations to increase the contrast between the two average error values, and therefore to reduce the number of runs necessary. Mean values and other statistical data are given in Table 1.

For N_p equal to 48 points and more, the average improvement was consistently about 20 percent (Fig. 2) with 90 percent of runs showing improvement as a result of applying the weighted least-squares solution.

For this simulation, the standard deviation of the measurement error, σ_{me}, is constant throughout the workspace. At the same time, the variance of the joint variable error's contribution to the pose error varies as a function of the robot's pose [Eq. (10)]. Since the measurement variance is added to the joint variable noise variance to obtain weights in the $\mathbf{V}$ matrix, the relative difference between the weights decreases in our simulation when the measurement error is increased. As a result, the effectiveness of the weighted least-squares also

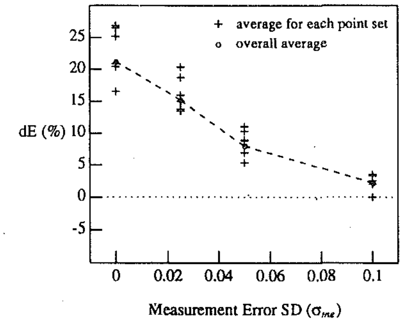

Fig. 3 Relative decrease in position error as a function of the measurement error

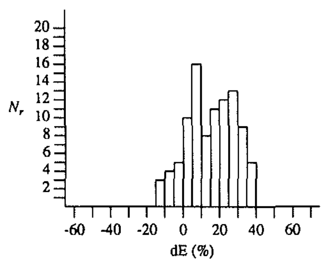

Fig. 4(*a*) *dE* distribution: 0.025 mm measurement error (N_p = 48)

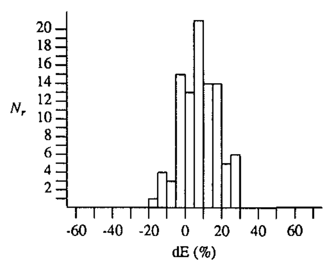

Fig. 4(*b*) *dE* distribution: 0.05 mm measurement error (N_p = 48)

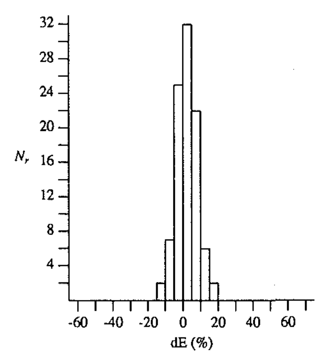

Fig. 4(*c*) *dE* distribution: 0.1 mm measurement error (N_p = 48)

would be expected to show a decrease with larger measurement error.

The significance of this effect will depend on the characteristics of the particular robot and the measurement system used. Note however that while some measurement devices have constant measurement error variance throughout the workspace (e.g., coordinate measuring machines) this is not the case for others (e.g., theodolite targeting). In the latter case, the contribution of the weighted least-squares method may be even greater than that obtained here.

To investigate the effect of increasing measurement error in this simulation, tests were conducted where the value of the measurement error standard deviation σ_{me} was increased from 0 to 0.1 mm while keeping the number of measurement points constant at $N_p = 48$. As expected, Fig. 3 shows the decrease of the mean improvement from 21.2 percent for the σ_{me} equal zero to about 2.2 percent for σ_{me} of 0.1 mm. Also, Figs. 1(*c*) and 4(*a–c*) illustrate how with increase of σ_{me} the standard deviation of the sample data decreases from about 15.2 to 5.5 (Table 2). This can be explained by the decreasing influence of the weighting matrix as the differences between its elements decrease.

Table 2 Improvement (dE) as a function of measurement error σ_{me}(N_p = 48)

σ_{me} (mm)	MEAN	SD	LOWEST	HIGHEST	% ABOVE ZERO
0.0	21.24	15.16	-35.7	54.0	91.7
0.025	15.22	13.15	-14.5	39.9	87.5
0.05	8.00	10.16	-16.7	28.8	76.0
0.1	2.22	5.49	-14.8	19.1	64.6

Alternative weighting functions of the error variance inverses were tried for comparison with a weighting matrix which is proportional to the inverse of **V**. The results were either worse or showed no significant improvement which supports the concept of applying the Gauss-Markov Theorem to this problem.

5 Conclusions

The Gauss-Markov Theorem was applied to the task of kinematic parameter estimation in robot calibration. Simulation demonstrated that a 20 percent decrease in average position error of the calibrated robot is possible as a result of substituting a weighted least-squares for the ordinary least-squares solution method.

The magnitude of the expected improvement will be determined by the particular circumstances of each calibration. The greater the variation of the random error variance over the measurement points used, the higher benefits can be expected from the application of GMT. In addition, the achievable improvement will depend on the accuracy of the estimate of **V**.

The application of the weighted least-squares would be expected to be particularly useful where a large number of similar robots are to be calibrated. Then, it becomes worthwhile to gather information about the random error behavior of these robots in order to obtain an accurate estimate of the weighting matrix. This would lead to the reduction of the required number of measurements, and thus savings of time and resources.

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